International Journal of Theoretical Physics, Vol. 33, No. 3, 1994

Spin-Gauge Theory of Gravity with Higgs-Field Mechanism

H. Dehnen[1] and E. Hitzer[1]

Received October 27, 1992

We propose a Lorentz-covariant Yang–Mills spin-gauge theory, where the function-valued Dirac matrices play the role of a nonscalar Higgs-field. As symmetry group we choose $SU(2) \times U(1)$. After symmetry breaking a nonscalar Lorentz-covariant Higgs-field gravity appears, which can be interpreted within a classical limit as Einstein's metrical theory of gravity, where we restrict ourselves in a first step to its linearized version.

1. INTRODUCTION

Within the solar system and the binary pulsar PSR 1913 + 16 the classical gravitational interaction is described very well by Einstein's general relativity. However, this theory—simultaneously the oldest non-Abelian gauge theory with the Poincaré group as gauge group—has not been quantizable. On the other hand, all the other fundamental interactions and their unifications are described successfully by quantizable Lorentz-covariant gauge theories with unitary gauge groups. Therefore, the suspicion exists that Einstein's theory represents only a classical macroscopic description of gravity and that the fundamental microscopic gravitational interaction between elementary particles is also described by a unitary gauge group on the Minkowski space-time in such a way that Einstein's theory of macroscopic gravity is reached as an effective theory within a certain classical limit, similarly as in the strong interaction the nuclear forces follow from the quantum chromodynamics.[2] In this way the problem of quantization of gravity and its unification with the other interactions would be solvable.

[1]Physics Department, University of Konstanz, Box 5560, M677, Konstanz, Germany.
[2]For this general intention see also Stumpf (1988).

In this connection the statement is of interest (Dehnen *et al.*, 1990) that the scalar Higgs field of elementary particle physics, the basis of which are of course unitary transformation groups, mediates a Lorentz-invariant attractive gravitational interaction between those elementary particles which become massive by the spontaneous symmetry breaking, i.e., the Higgs field has its source only in the mass and acts back only on the mass of the particles. The equivalence of inertial and gravitational mass is fulfilled automatically within this Higgs gravity. But if the strength of this gravity shall be of the order of the Newtonian one, the mass of the gauge bosons will be of the order of the Planck mass.

For the last reason the standard Higgs gravity, e.g., within the electroweak interaction (Dehnen and Frommert, 1991), has presumably nothing to do with usual gravity. However, here the question arises whether Einstein's tensorial gravity may be a consequence of a more sophisticated Higgs field, which is especially not a scalar one.

For this we extend back to a Yang–Mills $SU(2) \times (1)$ spin-gauge theory of gravity on the Minkowski space-time of special relativity proposed by Dehnen *et al.* (Dehnen and Ghaboussi, 1985, 1986; see also Chisholm and Farwell, 1989). In this theory, where a subgroup of the unitary transformations of Dirac's γ-matrices between their different representations [internal spin group (see also Drechsler, 1988; Bade and Jehle, 1953; cf. also Laporte and Uhlenbeck, 1931; Barut and McEwan, 1984)] is gauged, the γ-matrices became function-valued, but remained covariantly constant with respect to the internal spin group, whereas the gravitational interaction is mediated by the four gauge bosons belonging to the group $SU(2) \times (1)$ and the classical non-Euclidian metric is constructed out of them as an effective field in a certain manner.

Here a modification in the sense of the Higgs-field gravity is indicated: Instead of considering the γ-matrices as covariantly constant, it is possible to treat them as true field variables with a Higgs–Lagrange density, because also the γ-matrices possess a nontrivial ground state, namely the usual constant standard representations. Because the γ-matrices can be understood as square root of the metric, the gauge group is that of the square root of the metric; moreover, in consequence of this group the several spin states (or particle–antiparticle states) are indistinguishable with respect to the interaction following from gauging the spin group. Both properties suggest that real gravity is involved.

In this way we get a quantizable unitary spin-gauge theory with Dirac's γ-matrices as Higgs fields; on this level a unification with all the other interactions may be possible. After spontaneous symmetry breaking a nonscalar Higgs gravity appears which can be identified in a classical limit with Einstein's gravity, where we restrict ourselves in the first step for simplicity to the linear theory. The essential points are the following: the theory is from the beginning only Lorentz covariant. After symmetry breaking and performing a unitary gauge, the action of the excited γ-Higgs field on the fermions in the Minkowski space-time is reinterpreted as if there would exist non-Euclidian space-time connections and a non-Euclidian metric (effective metric), in which the fermions move freely; then the deviation from the Minkowski space-time describes classical gravity. This happens, as usual, in the de Donder gauge and not in general coordinate covariance, which depends also on the fact that with the choice of the unitary gauge a gauge fixing is connected. In this way the gravitational constant is produced only by the symmetry breaking and the non-Euclidian metric comes out to be an effective field, whereas the gauge bosons get masses of the order of the Planck mass and can be therefore neglected in the low-energy limit; but in the high-energy limit ($\simeq 10^{19}$ GeV) an additional "strong" gravitational interaction exists. Simultaneously, our results give a new light on the role of the Higgs mechanism.

Finally, we note that, as in the previous spin-gauge theory (Ghaboussi *et al.*, 1987) a richer space-time geometrical structure results than only a Riemannian one. We find also torsion, which can be neglected, however, in the classical limit, and nonmetricity. The question of whether it is possible to change the Lagrangian so that nonmetricity does not appear will be clarified in a later paper.

2. THE MODEL

In the beginning we repeat briefly the foundations of the previous work (Dehnen and Ghaboussi, 1986; see also Babu Joseph and Sabir, 1988) so far as necessary. Using 4-spinors, it is appropriate to introduce the transformation matrices of the group $SU(2)\times U(1)$ in their 4×4 representation ($a=0, 1, 2, 3$):

$$U = e^{i\lambda_a(x^\mu)\tau^a} \tag{2.1}$$

where the $SU(2)$ generators are given by the Pauli matrices σ^i as follows ($i=1, 2, 3$),[3]

$$\tau^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \tag{2.2}$$

The $U(1)$ generator τ^0 may be diagonal and commutes with (2.2); but its special form shall be determined only later. Thus the commutator relations

[3]The explicit form of (2.2) is only used in (4.7).

for the generators τ^a are

$$[\tau^b, \tau^c] = i\epsilon_a{}^{bc}\tau^a \tag{2.3}$$

where $\epsilon_a{}^{bc}$ is the Levi-Civita symbol with the additional property to be zero if a, b or c is zero.

Then the 4-spinor ψ and the Dirac matrices γ^μ transform as[4]

$$\psi' = U\psi, \qquad \gamma'^\mu = U\gamma^\mu U^{-1} \tag{2.4}$$

and the covariant spinor derivative reads

$$D_\mu \psi = (\partial_\mu + ig\omega_\mu)\psi \tag{2.5}$$

(g gauge coupling constant). The gauge potentials ω_μ obey the transformation law[5]

$$\omega'_\mu = U\omega_\mu U^{-1} + \frac{i}{g} U_{|\mu} U^{-1} \tag{2.6}$$

and are connected with the real-valued gauge fields $\omega_{\mu a}$ by

$$\omega_\mu = \omega_{\mu a}\tau^a \tag{2.7}$$

According to (2.4), Dirac's γ-matrices become necessarily function-valued, in consequence of which we need determination equations for them; as such ones we have chosen in our previous paper in analogy to general relativity

$$D_\alpha \gamma^\mu = \partial_\alpha \gamma^\mu + ig[\omega_\alpha, \gamma^\mu] = 0, \qquad \gamma^{(\mu}\gamma^{\nu)} = \eta^{\mu\nu} \cdot \mathbf{1} \tag{2.8}$$

[$\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ Minkowski metric]. Because the γ-matrices are the formal square root of the metric, the gauge transformations (2.4) are those which are associated with the root of the metric. Therefore the concept described by the formulas (2.1)–(2.7) may have something to do with gravity. And indeed, in our previous paper we could show that a space-time geometrical interpretation of the theory results in an effective non-Euclidian metric given by

$$g_{\mu\nu} = \omega_{\mu a}\omega_{\nu b}\eta^{ab} \tag{2.9}$$

However, the result (2.9) is connected with the condition that the gauge potentials $\omega_{\mu a}$ never vanish and possess a nontrivial ground-state representing according to (2.9) in the lowest order the Minkowski metric.

[4]γ^μ are tensors with respect to the unitary transformations (2.1), but they are not elements of the adjoint representation.

[5]$|\mu$ denotes the partial derivative with respect to the coordinate x^μ.

This is an unusual feature; furthermore the conditions (2.8) are chosen for simplicity. Therefore it may be justified to give up the relations (2.8) and (2.9) and to consider Dirac's γ-matrices as true field variables with a Higgs–Lagrange density, so that the nontrivial ground state can be identified with the constant standard representations. It will come out that after symmetry breaking the excited γ-Higgs fields mediate a nonscalar Higgs gravity, which results finally in Einstein's metrical theory, where instead of (2.9) the connection between the effective non-Euclidian metric and the γ-Higgs field will be deduced from a space-time geometrical interpretation of the equation of motion for the 4-momentum of the fermions described by the spinor fields ψ.

3. LAGRANGE DENSITY AND FIELD EQUATIONS

The translation of the model into a field-theoretic description results in a Lagrange density consisting of three minimally coupled Lorentz- and gauge-invariant real-valued parts ($\hbar = 1, c = 1$):

$$\mathscr{L} = \mathscr{L}_M(\psi) + \mathscr{L}_F(\omega) + \mathscr{L}_H(\gamma) \tag{3.1}$$

Beginning with the last part, $\mathscr{L}_H(\gamma)$ belongs to the γ-Higgs field and has the form

$$\mathscr{L}_H(\gamma) = \frac{1}{2}\operatorname{tr}[(D_\alpha \tilde{\gamma}^\mu)(D^\alpha \tilde{\gamma}_\mu)] - V(\tilde{\gamma}) - k\bar{\psi}(N^\dagger \hat{x} + \hat{x}^\dagger N)\psi \tag{3.2}$$

where

$$V(\tilde{\gamma}) = \frac{\mu^2}{2}\operatorname{tr}(\tilde{\gamma}^\mu \tilde{\gamma}_\mu) + \frac{\lambda}{4!}(\operatorname{tr}\tilde{\gamma}^\mu \tilde{\gamma}_\mu)^2 \tag{3.2a}$$

is the Higgs potential. Herein $\tilde{\gamma}^\mu$ denotes from now on the dynamic function-valued γ-matrices, which obey the transformation law (2.4) and the ground states of which are proportional to the constant standard representations γ^μ (bear this change of notation in mind). The last term on the right-hand side of (3.2) represents the Yukawa coupling term for generating the mass of the fermions; this should result from a standard isospin-valued scalar Higgs field Φ, e.g., from that of the electroweak interaction, with the ground-state unit isovector N ($N^\dagger N = 1$; $\hat{x}$ is the Yukawa coupling matrix). Of course the dynamic parts of this Higgs field Φ as well as of the other interactions are neglected in (3.1) and (3.2) because we restrict our consideration to the gravitational aspects only.

At this point one could think of substituting the usual Higgs field Φ by the γ-Higgs field. But this leads later to problems in connection with a purely geometrical interpretation of the forces. Therefore we retain Φ at least in the first step.

The second term on the right-hand side of (3.1) is that of the gauge fields ω_μ:

$$\mathscr{L}_F(\omega) = -\frac{1}{16\pi} F_{\mu\nu a} F^{\mu\nu}{}_b s^{ab} \tag{3.3}$$

where s^{ab} is the group metric of $SU(2) \times U(1)$ and can be taken here as δ^{ab} (but compare the previous work). The gauge field strengths are defined in the usual manner by

$$\mathscr{F}_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = F_{\mu\nu a}\tau^a \tag{3.4}$$

with

$$F_{\mu\nu a} = \omega_{\nu a|\mu} - \omega_{\mu a|\nu} - g\epsilon_a{}^{jk}\omega_{\mu j}\omega_{\nu k} \tag{3.4a}$$

The first Lagrangian in (3.1) concerns the fermionic matter fields and takes the form [ψ is only proportional to the Dirac spinor, see (4.4), but can have arbitrary degrees of isospin freedom]:

$$\mathscr{L}_M(\psi) = \frac{i}{2}\bar{\psi}\tilde{\gamma}^\mu D_\mu\psi - \frac{i}{2}\overline{(D_\mu\psi)}\tilde{\gamma}^\mu\psi \tag{3.5}$$

The adjoint spinor $\bar{\psi}$ is given by

$$\bar{\psi} = \psi^\dagger\zeta \tag{3.6}$$

wherein ζ represents the $SU(2) \times U(1)$-covariant matrix with the property

$$(\zeta\tilde{\gamma}^\mu)^\dagger = \zeta\tilde{\gamma}^\mu \tag{3.7}$$

In view of the commutability of covariant derivative and multiplication with ζ in (3.5) it is further necessary that

$$D_\mu\zeta = 0 \tag{3.8}$$

So long as (see Section 4)

$$[\gamma^0, \tau^a] = 0 \tag{3.9}$$

equations (3.7) and (3.8) will be fulfilled only (up to a constant factor) by

$$\zeta = \gamma^0 \qquad (\zeta^\dagger = \zeta, \zeta^2 = 1) \tag{3.10}$$

so that (3.6) yields as usual $\bar{\psi} = \psi^\dagger\gamma^0$. Because of (3.9) the matrix ζ is not only covariant, but even invariant under gauge transformations. These results depend essentially on the relation (3.9), which may be not valid in a larger group [e.g., $U(4)$].[6] Finally we note that one can prove easily with the use of (4.8) that all three expressions (3.2), (3.3), and (3.5) of the Lagrangian are real-valued and contain no dimensional parameter with exception of k and μ^2 in (3.2), which have the dimension of a mass squared.

The field equations following from the action principle associated with (3.1) are given by the generalized Dirac equation

$$i\tilde{\gamma}^\mu D_\mu \psi + \frac{i}{2}(D_\mu \tilde{\gamma}^\mu)\psi - k(N^\dagger \hat{x} + \hat{x}^\dagger N)\psi = 0 \tag{3.11}$$

as well as its adjoint equation, by the inhomogeneous Yang–Mills equation

$$\partial_\nu F^{\nu\mu}{}_a + g\epsilon_a{}^{bc} F^{\nu\mu}_b \omega_{\nu c} = 4\pi j^\mu_a \tag{3.12}$$

with the gauge currents

$$j^\mu_a = j^\mu_a(\psi) + j^\mu_a(\gamma) = \frac{g}{2}\bar{\psi}\{\tilde{\gamma}^\mu, \tau_a\}\psi + ig\ \mathrm{tr}([\tilde{\gamma}^\alpha, \tau_a]D^\mu \tilde{\gamma}_\alpha) \tag{3.12a}$$

belonging to the matter and the Higgs field, respectively, and by the γ-Higgs field equation:[7]

$$D_\alpha D^\alpha \tilde{\gamma}^\mu{}_A{}^B - \frac{i}{2}[\bar{\psi}^B \cdot (D^\mu \psi)_A - (\overline{D^\mu \psi})^B \cdot \psi_A] + \left[\mu^2 + \frac{\lambda}{6}\mathrm{tr}(\tilde{\gamma}^\alpha \tilde{\gamma}_\alpha)\right]\tilde{\gamma}^{\mu B}_A = 0 \tag{3.13}$$

Herein the lower capital Latin index A and the upper index B denote the contragradiently transformed rows and columns of the spinorial matrices, respectively. The homogeneous Yang–Mills equation following from the Jacobi identity reads

$$\partial_{[\mu} F_{\nu\lambda]a} + g\omega_{k[\mu} F_{\nu\lambda]j} \epsilon^{kj}{}_a = 0 \tag{3.14}$$

Finally we note the conservation laws valid modulo the field equations. First, from (3.12) the gauge current conservation follows immediately:

$$\partial_\mu \left(j^\mu{}_a + \frac{g}{4\pi}\epsilon_a{}^{bc} F^{\mu\nu}{}_b \omega_{\nu c} \right) = 0 \tag{3.15}$$

[6] A generalization of the theory to the full gauge group $U(4)$ is in preparation (see also Drechsler (1988)).

[7] If $\tilde{\gamma}^\mu$ is considered to be traceless [see (4.2) and (4.8)], then also the traceless version of (3.13) is valid only.

Second, the energy-momentum law takes the form

$$\partial_\nu T_\mu{}^\nu = 0 \tag{3.16}$$

where $T_\mu{}^\nu$ is the gauge-invariant canonical energy-momentum tensor consisting of three parts corresponding to (3.1)

$$T_\mu{}^\nu = T_\mu{}^\nu(\psi) + T_\mu{}^\nu(\omega) + T_\mu{}^\nu(\gamma) \tag{3.17}$$

with (modulo Dirac equation)

$$T_\mu{}^\nu(\psi) = \frac{i}{2}[\bar{\psi}\tilde{\gamma}^\nu D_\mu \psi - (\overline{D_\mu \psi})\tilde{\gamma}^\nu \psi] \tag{3.18a}$$

$$T_\mu{}^\nu(\omega) = -\frac{1}{4\pi}\left[F_{\mu\alpha a}F^{\nu\alpha a} - \frac{1}{4}F_a^{\alpha\beta}F^a_{\alpha\beta}\delta_\mu^\nu\right] \tag{3.18b}$$

$$\begin{aligned} T_\mu{}^\nu(\gamma) &= \mathrm{tr}[(D^\nu\tilde{\gamma}_\alpha)(D_\mu\tilde{\gamma}^\alpha)] \\ &\quad - \delta_\mu^\nu\left\{\frac{1}{2}\, tr[(D_\alpha\tilde{\gamma}^\beta)(D^\alpha\tilde{\gamma}_\beta)] - \frac{\mu^2}{2}\mathrm{tr}(\tilde{\gamma}^\alpha\tilde{\gamma}_\alpha) - \frac{\lambda}{4!}(\mathrm{tr}\,\tilde{\gamma}^\alpha\tilde{\gamma}_\alpha)^2\right\} \end{aligned} \tag{3.18c}$$

Because of the Yukawa coupling term in (3.2), the trace of (3.18a) does not vanish. With the use of the Dirac equation (3.11) and its adjoint equation one finds

$$T_\mu{}^\mu(\psi) = k\bar{\psi}[N^\dagger\hat{x} + \hat{x}^\dagger N]\psi \tag{3.19}$$

where the bracket represents the fermionic mass matrix.

By insertion of (3.18) into (3.17) one obtains from (3.16) the equation of motion for the fermions. After substitution of the second covariant derivatives of the γ-Higgs field using the field equation (3.13), one finds with the help of the Yang–Mills equations (3.12) and (3.14)

$$\partial_\nu T^{\mu\nu}(\psi) = -\frac{i}{2}[\bar{\psi}(D^\mu\tilde{\gamma}^\alpha)(D_\alpha\psi) - (\overline{D_\alpha\psi})(D^\mu\tilde{\gamma}^\alpha)\psi] + F^\mu{}_{\alpha a}j^{\alpha a}(\psi) \tag{3.20}$$

Integration over the spacelike hypersurface $t = \mathrm{const}$ and neglect of surface integrals in the spacelike infinity yield the momentum law for the 4-momentum $p^\mu = \int T^{\mu 0}(\psi)\, d^3x$ of the fermions. On the right-hand side of (3.20) one recognizes the Lorentz forces of the gauge fields and the force of the γ-Higgs field.

We finish with two remarks. First, the energy-momentum tensor $T_\mu{}^\nu(\gamma)$, equation (3.18c), does not vanish for the ground state [see (4.2)], but has the value

$$\overset{(0)}{T}_\mu{}^\nu(\overset{(0)}{\gamma}) = -\frac{3}{2}\frac{\mu^4}{\lambda}\delta_\mu{}^\nu \tag{3.21}$$

However, this can be renormalized to zero by changing the Higgs potential

(3.2a) correspondingly; otherwise (3.21) will give rise within the complete theory to a cosmological constant. Second, the γ-Higgs field equation (3.13) contains as source for $\tilde{\gamma}^{\mu}$ the fermionic energy-momentum tensor $T_{\mu}{}^{\nu}(\psi)$ in its spinor valued form; and in this form it appears also in the γ-Higgs field force of (3.20). This fact confirms the supposition that the γ-Higgs field equation results in Einstein's field equation of gravitation (for the fermions) after a space-time geometrical interpretation of the γ-Higgs field forces in (3.20) defining the effective space-time geometrical connection coefficients.

4. SPONTANEOUS SYMMETRY BREAKING

Although one can recognize the gravitational structure already in equations (3.13) and (3.20), the space-time geometrical interpretation is only possible after symmetry breaking. The minimum of the energy-momentum tensor (3.18) in the absence of matter and gauge fields is reached when the Higgs potential (3.2a) is in its minimum defined by

$$\operatorname{tr}\left(\overset{(0)}{\tilde{\gamma}}{}^{\mu}\overset{(0)}{\gamma}{}_{\mu}\right) = -\frac{6\mu^2}{\lambda} = v^2 \qquad (\mu^2 < 0) \tag{4.1}$$

The ground-state $\overset{(0)}{(\tilde{\gamma})}{}^{\mu}$ of the γ-Higgs field must be proportional to the (constant) Dirac standard representation γ^{μ}, i.e.,[8]

$$\overset{(0)}{(\tilde{\gamma})}{}^{\mu} = b\gamma^{\mu}$$

Insertion into (4.1) results, because of $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}\cdot\mathbf{1}$ in $b = v/4$, so that we have for the ground state

$$\overset{(0)}{\tilde{\gamma}}{}^{\mu} = \frac{v}{4}\gamma^{\mu} \tag{4.2}$$

Here the Lagrange density (3.5) for the spinorial matter fields reads, considering the γ-Higgs field ground state only,

$$\frac{i}{2}\bar{\psi}\frac{v}{4}\gamma^{\mu}\partial_{\mu}\psi + \text{h.c.} \tag{4.3}$$

Comparison with the usual Dirac Lagrangian $(i/2)\bar{\psi}_{\text{DIR}}\gamma^{\mu}\partial_{\mu}\psi_{\text{DIR}}$ results in (ψ_{DIR} Dirac spinor)

$$\psi = \frac{2}{\sqrt{v}}\psi_{\text{DIR}} \tag{4.4}$$

[8]Of course, global unitary transformations between the different standard representations and simultaneously of the generators are allowed.

Here the fermionic mass term in (3.2), identical with the trace (3.19) of the energy-momentum tensor $T_\mu{}^\nu(\psi)$, takes the form

$$T_\mu{}^\mu(\psi_{\mathrm{DIR}}) = \bar{\psi}_{\mathrm{DIR}}\hat{m}\psi_{\mathrm{DIR}} \tag{4.5}$$

with the mass matrix

$$\hat{m} = m(N^\dagger\hat{x} + \hat{x}^\dagger N), \qquad m = \frac{4k}{v} \tag{4.5a}$$

On the other hand, the Higgs-field gauge current $j^\mu_a(\gamma)$ gives rise after symmetry breaking to the mass of the gauge bosons ω^μ_a. In the lowest order we find from (3.12a) with the use of (4.2)

$$\begin{aligned} -4\pi j^\mu_a(\overset{0}{\gamma}) &= M^2_{ab}\omega^{\mu b}, \qquad M^2_{ab} = M^2_{ab}{}^{\rho\nu}\eta_{\rho\nu} \\ M^2_{ab}{}^{\rho\nu} &= -\frac{\pi}{4}g^2v^2\,\mathrm{tr}([\tau_a,\gamma^\rho][\tau_b,\gamma^\nu]) \end{aligned} \tag{4.6}$$

Here it is convenient to choose the $U(1)$ generator explicitly. If we take the unit matrix, the gauge boson $\omega^\mu{}_0$ remains massless of course (rest symmetry) and must be taken into account also in the low-energy limit. In order to avoid this,[9] the only other possibility is the choice $\tau^0 = \frac{1}{2}\gamma^0$. Doing this, we obtain from (4.6) with the use of (2.2) the diagonal mass matrix for the gauge bosons:

$$\begin{aligned} M^2_{00} &= 3\pi g^2v^2 \\ M^2_{ij} &= 2\pi g^2v^2\delta_{ij} \end{aligned} \tag{4.7}$$

and zero otherwise. As we will see later, the value of (4.7) is of the order of the square of the Planck mass ($\triangleq 10^{19}$ GeV).

As one can prove easily, the general Higgs field $\tilde{\gamma}^\mu$ can be represented by

$$\tilde{\gamma}^\mu(x^\alpha) = h^\mu{}_\lambda(x^\nu)U\overset{(0)}{\gamma}{}^\lambda U^{-1} \tag{4.8}$$

so that it can be reduced within the unitary gauge as usual to the ground state (4.2) in the following way:

$$\tilde{\gamma}^\mu(x^\nu) = h^\mu{}_\lambda(x^\nu)\overset{(0)}{\gamma}{}^\lambda \tag{4.8a}$$

where[10]

$$h^\mu{}_\lambda(x^\nu) = \delta^\mu{}_\lambda + \epsilon^\mu{}_\lambda(x^\nu) \tag{4.8b}$$

[9]It seems to us not suitable to identify this boson with the photon, in view of the electroweak interaction.

[10]Lifting and lowering of indices is performed always with $\eta^{\mu\nu}$ and $\eta_{\nu\lambda}$, respectively.

and $\epsilon^{\mu}{}_{\lambda}(x^{\nu})$ describes the deviations from the ground state, i.e., the excited Higgs field. Here we are able to write down all field equations after symmetry breaking exactly in a non-matrix-valued form. Of course, the Lorentz-tensor $h^{\mu}{}_{\lambda}(x^{\nu})$ looks like a tetrad field, but its determination and connection with the effective non-Euclidian metric follow only from the γ-Higgs field equation after symmetry breaking.

5. FIELD EQUATIONS AFTER SYMMETRY BREAKING AND GRAVITATIONAL INTERACTION

In this section we restrict ourselves in a first step for simplicity to the linearized theory, i.e., $|\epsilon^{\mu}{}_{\lambda}| \ll 1$ (weak-field limit). We start in view of the gravitational aspect with the Higgs field equation (3.13). Going over from a spinorial description to a Lorentz-tensorial equation, we multiply (3.13) at first by $\gamma^{\lambda}{}_{B}{}^{A}$. Then after insertion of (4.2), (4.4), (4.8a), and (4.8b) we obtain, linearized in $\epsilon^{\mu}{}_{\lambda}$ under neglect of the gauge-boson interaction because of (4.7) (low-energy limit)

$$\partial_{\alpha}\partial^{\alpha}\epsilon^{\mu\lambda} - \frac{\mu^2}{2}\epsilon\eta^{\mu\lambda} = \frac{4}{v^2} T^{\mu\lambda}(\psi_{\text{DIR}}) \tag{5.1}$$

where

$$T^{\mu\lambda}(\psi_{\text{DIR}}) = \frac{i}{2}[\bar{\psi}_{\text{DIR}}\gamma^{\lambda}D^{\mu}\psi_{\text{DIR}} - (\overline{D^{\mu}\psi}_{\text{DIR}})\gamma^{\lambda}\psi_{\text{DIR}}] \tag{5.1a}$$

is the usual (canonical) Dirac energy-momentum tensor. Obviously the antisymmetric and the traceless symmetry part of $\epsilon^{\mu\lambda}$ remain massless, whereas the trace $\epsilon = \epsilon^{\mu\lambda}\eta_{\mu\lambda}$ possesses the Higgs mass:

$$M = (-2\mu^2)^{1/2} \tag{5.1b}$$

Furthermore, if (5.1) is to describe usual gravity, $v^2 \sim G^{-1}$ (G is the Newtonian gravitational constant) must be valid, so that (4.7) is indeed of the order of the square of the Planck mass $M_{\text{Pl}} = 1/\sqrt{G}$.

Before comparing (5.1) with Einstein's field equations it is appropriate to interpret first the Higgs field forces in (3.20) geometrically, where in the low-energy limit the Lorentz forces of the gauge fields can be neglected. Insertion of (4.2), (4.4), (4.8a), and (4.8b) into (3.20) gives with respect to (3.18a) and (5.1a)

$$\partial_{\nu} T^{\mu\nu}(\psi_{\text{DIR}}) = -\partial^{\nu}\epsilon_{\nu\rho} T^{\mu\rho}(\psi_{\text{DIR}}) - \partial^{\mu}\epsilon_{\rho\nu} T^{\rho\nu}(\psi_{\text{DIR}}) \tag{5.2}$$

linearized with regard to $\epsilon^{\mu}{}_{\lambda}$. The equations (5.1) and (5.2) describe the γ-Higgs field interaction in its linearized version, which is obviously very similar to that of general relativity.

Now, the comparison of (5.2) with the energy-momentum law of a classical affine geometrical theory with the affine connections $\Gamma^{\mu}{}_{\nu\rho}$

$$D_{\nu}^{(\Gamma)} T^{\mu\nu} = 0 \Rightarrow \partial_{\nu} T^{\mu\nu} = -\Gamma^{\nu}{}_{\nu\rho} T^{\mu\rho} - \Gamma^{\mu}{}_{\nu\rho} T^{\rho\nu} \tag{5.3}$$

results in the identification

$$\Gamma^{\mu}{}_{\nu\rho} = \partial^{\mu}\epsilon_{\rho\nu}; \qquad \Gamma^{\nu}{}_{\nu\rho} = \partial^{\nu}\epsilon_{\nu\rho} \tag{5.4}$$

These two relations are only compatible if

$$\partial^{\nu}\epsilon_{[\rho\nu]} = \Gamma^{\nu}{}_{[\nu\rho]} = 0 \Rightarrow \partial^{\nu}\epsilon_{\rho\nu} = \partial^{\nu}\epsilon_{(\rho\nu)} \tag{5.4a}$$

is valid. Here the forces of the excited γ-Higgs field on the fermions in the Minkowski space-time are reinterpreted as the action of non-Euclidian space-time geometrical connections.

Consequently, in the space-time geometrical limit the excited Higgs field $\epsilon^{\mu}{}_{\lambda}$, or more precisely its derivatives, play effectively the role of affine connections (effective connections). Their field equations are obtained in the following way: With the identity (5.4) the equations (5.1) take the form, assuming a negligible Higgs mass (5.1b),

$$\partial_{\alpha}\partial^{\alpha}\epsilon_{(\mu\nu)} = \partial_{\alpha}\Gamma^{\alpha}{}_{(\mu\nu)} = \frac{4}{v^2} T_{(\mu\nu)}(\psi_{\mathrm{DIR}}) \tag{5.5a}$$

and

$$\partial_{\alpha}\partial^{\alpha}\epsilon_{[\mu\nu]} = \partial_{\alpha}\Gamma^{\alpha}{}_{[\nu\mu]} = \frac{4}{v^2} T_{[\mu\nu]}(\psi_{\mathrm{DIR}}) \tag{5.5b}$$

In the lowest order, which is considered here only, the right-hand sides of (5.5) possess in view of (5.2) vanishing divergences. Therefore the following constraints hold in consequence of the field equations:

$$\left.\begin{aligned} \partial^{\nu}\epsilon_{(\mu\nu)} &= 0 \qquad (5.6a)\\ \partial^{\nu}\epsilon_{[\mu\nu]} &= 0 \qquad (5.6b)\end{aligned}\right\} \Rightarrow \partial^{\nu}\epsilon_{\mu\nu} = 0$$

the second of which guarantees the fulfillment of the compatibility condition (5.4a).

Evidently the field equation (5.5b) represents the equation of the effective torsion. Its source is the antisymmetric part of the fermionic energy-momentum tensor (5.1a), which can be written with the use of the Dirac equation in the lowest order:

$$T_{[\mu\nu]}(\psi_{\mathrm{DIR}}) = \frac{1}{2}\left[\bar{\psi}_{\mathrm{DIR}}\gamma_{[\mu}\sigma_{\nu]}{}^{\lambda}D_{\lambda}\psi_{\mathrm{DIR}} + (\overline{D_{\lambda}\psi_{\mathrm{DIR}}})\sigma^{\lambda}{}_{[\mu}\gamma_{\nu]}\psi_{\mathrm{DIR}}\right] \tag{5.7}$$

where $\sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]}$ is the spin operator. In consequence of this the torsion has its origin in the fermionic spin density (cf. also Hehl, 1973, 1974).

Therefore, if we neglect in the classical macroscopic limit all spin influences, the solution of (5.5b) is

$$\epsilon_{[\mu\nu]} \equiv 0, \qquad \Gamma^{\alpha}{}_{[\mu\nu]} \equiv 0 \tag{5.8}$$

whereby the compatibility condition (5.4a) is fulfilled identically.

For discussion of the field equation (5.5a) for the symmetric part of the effective connections we compare directly with Einstein's linearized field equations of gravity. Setting

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{5.9}$$

and choosing the de Donder gauge

$$\partial^{\nu}\left(\gamma_{\mu\nu} - \frac{1}{2}\gamma\eta_{\mu\nu}\right) = 0 \tag{5.9a}$$

($\gamma = \gamma_{\mu\nu}\eta^{\mu\nu}$), we find that

$$\partial_{\alpha}\partial^{\alpha}\frac{1}{2}\left(\gamma_{\mu\nu} - \frac{1}{2}\gamma\eta_{\mu\nu}\right) = -8\pi G T_{(\mu\nu)} \tag{5.10}$$

The comparison with (5.5a) results immediately in

$$\begin{aligned} \epsilon_{(\mu\nu)} &= -\frac{1}{2}\alpha\left(\gamma_{\mu\nu} - \frac{1}{2}\gamma\eta_{\mu\nu}\right), \\ \gamma_{\mu\nu} &= -\frac{2}{\alpha}\left(\epsilon_{(\mu\nu)} - \frac{1}{2}\epsilon\eta_{\mu\nu}\right) \end{aligned} \qquad \gamma = \frac{2}{\alpha}\epsilon \tag{5.11}$$

and

$$v = (2\pi\alpha G)^{-1/2} \tag{5.12}$$

up to the proportional constant α. Consequently the constraint (5.6a) is identical with the de Donder condition (5.9a) and Newton's gravitational constant G is correlated, as expected, with the Higgs-field ground-state value v. The constant α is adjusted in such a way that the equation of motion (5.2) goes over in the lowest order into the Newtonian gravitational law; for this

$$\epsilon_{00} = -\Phi \Rightarrow \alpha = \frac{1}{2} \tag{5.12a}$$

(Φ is the Newtonian gravitational potential) must be valid in view of (5.11) with $\gamma_{\mu\nu} = 2\Phi\,\mathrm{diag}(1, 1, 1, 1)$ according to (5.10). Here the effective non-Euclidian metric takes the form with respect to (5.9)

$$g_{\mu\nu} = \eta_{\mu\nu}(1 + 2\epsilon) - 4\epsilon_{(\mu\nu)} \tag{5.13}$$

For analyzing the symmetric connections $\Gamma^{\alpha}_{(\mu\nu)}$ after the identification (5.11) we calculate the linearized Christoffel symbol $\{{}^{\alpha}_{\mu\nu}\}$ belonging to the metric (5.13); we find linearized in $\epsilon_{\mu\nu}$

$$\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} = 2\partial^{\alpha}\epsilon_{(\mu\nu)} - 2\eta^{\alpha\lambda}\left(\epsilon_{(\nu\lambda)|\mu} + \epsilon_{(\mu\lambda)|\nu} - \eta_{\lambda(\mu}\epsilon_{|\nu)} + \frac{1}{2}\eta_{\mu\nu}\epsilon_{|\lambda}\right) \tag{5.14}$$

Because of

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{(\mu\nu)} = \partial^{\alpha}\epsilon_{(\mu\nu)} \tag{5.15}$$

[see (5.4) and (5.8)] the nonmetricity tensor $Q_{\mu\nu\lambda}$ belonging to (5.13) reads (also linearized in $\epsilon_{\mu\nu}$)

$$\begin{aligned} Q_{\mu\nu\lambda} \equiv -D^{(\Gamma)}_{\mu} g_{\nu\lambda} &= -\partial_{\mu} g_{\nu\lambda} + \Gamma^{\alpha}_{\mu\nu} g_{\alpha\lambda} + \Gamma^{\alpha}_{\mu\lambda} g_{\nu\alpha} \\ &= 4\partial_{\mu}\left[\epsilon_{(\nu\lambda)} - \frac{1}{2}\epsilon\eta_{\nu\lambda}\right] + \partial_{\nu}\epsilon_{(\mu\lambda)} + \partial_{\lambda}\epsilon_{(\mu\nu)} \end{aligned} \tag{5.16}$$

With (5.15) and (5.16) equation (5.14) can be written as

$$\Gamma^{\alpha}{}_{\mu\nu} = \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + \frac{1}{2}\eta^{\alpha\lambda}(Q_{\mu\nu\lambda} + Q_{\nu\mu\lambda} - Q_{\lambda\mu\nu}) \tag{5.17}$$

which is exactly the form of the (linearized) symmetric affine connection (see Schouten, 1954) in the torsion-free case [cf. (5.8)]. Obviously the spin-gauge theory of gravity possesses in its classical macroscopic limit a larger geometrical structure than only the Riemannian one: Besides the Christoffel connection, also nonmetricity exists.

In relation to this fact the field equations for both connection coefficients are of interest. Because of the identification of (5.5a) and (5.10) Einstein's field equations are to be expected for the Christoffel symbols. For the trace of the Christoffel symbols we obtain from (5.14) without any condition

$$\begin{Bmatrix} \alpha \\ \mu\alpha \end{Bmatrix} = 2\epsilon_{|\mu} \tag{5.18}$$

On the other hand, it follows from (5.14) with the use of the constraint (5.6a) that

$$\partial_{\alpha}\partial^{\alpha}\epsilon_{(\mu\nu)} = \frac{1}{2}\partial_{\alpha}\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} - \epsilon_{|\mu|\nu} + \frac{1}{2}\partial_{\alpha}\partial^{\alpha}\epsilon\eta_{\mu\nu} \tag{5.19}$$

Here the second term on the right side can be substituted by (5.18) and the third term by the trace of (5.5a), giving

$$\partial_{\alpha}\partial^{\alpha}\epsilon = \frac{4}{v^2} T(\psi_{\mathrm{DIR}}) \tag{5.20}$$

Doing this and substituting the left-hand side of (5.5a) by (5.19) yields

$$\partial_\alpha \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} - \partial_\nu \begin{Bmatrix} \alpha \\ \mu\alpha \end{Bmatrix} = -R_{\mu\nu} = \frac{8}{v^2}\left[T_{(\mu\nu)}(\psi_{\mathrm{DIR}}) - \frac{1}{2} T(\psi_{\mathrm{DIR}})\eta_{\mu\nu} \right] \quad (5.21)$$

($R_{\mu\nu}$ is the Ricci tensor). This is indeed together with (5.12) Einstein's linearized field equation for the Christoffel symbols belonging to the metric (5.13).

For investigation of the nonmetricity part of the affine connections (5.17) we insert (5.16) and take the divergence; thus with the use of (5.6a) we obtain

$$\partial_\alpha \frac{1}{2} \eta^{\alpha\lambda}(Q_{\mu\nu\lambda} + Q_{\nu\mu\lambda} - Q_{\lambda\mu\nu}) = -2\epsilon_{|\mu|\nu} - \partial_\alpha \partial^\alpha(\epsilon_{(\mu\nu)} - \epsilon\eta_{\mu\nu}) \quad (5.22)$$

or in correspondence with (5.21)

$$\begin{aligned} &\partial_\alpha \frac{1}{2} \eta^{\alpha\lambda}(Q_{\mu\nu\lambda} + Q_{\nu\mu\lambda} - Q_{\lambda\mu\nu}) - \partial_\nu \frac{1}{2} \eta^{\alpha\lambda}(Q_{\mu\alpha\lambda} + Q_{\alpha\mu\lambda} - Q_{\lambda\mu\alpha}) \\ &= -\partial_\alpha \partial^\alpha(\epsilon_{(\mu\nu)} - \epsilon\eta_{\mu\nu}) = -\frac{4}{v^2}(T_{\mu\nu} - T\eta_{\mu\nu}) \end{aligned} \quad (5.23)$$

where (5.5a) is used. The practical consequences of the appearance of nonmetricity or even better its avoidance shall be investigated later.

Now we note the Dirac equation for gravitational interaction according to the spin-gauge theory as well as the Yang–Mills equation for the very massive gauge fields. From (3.11) it follows immediately after insertion of (4.2), (4.4), (4.5a), and (4.8a) under neglect of the gauge-boson/graviton interaction (linearizing of the interaction terms)

$$\begin{aligned} &i\gamma^\mu(\partial_\mu + ig\omega_{\mu\alpha}\tau^\alpha)\psi_{\mathrm{DIR}} + i\left[\epsilon^\lambda{}_\mu \partial_\lambda + \frac{1}{2}(\partial_\lambda \epsilon^\lambda{}_\mu) \right] \gamma^\mu \psi_{\mathrm{DIR}} \\ &\quad -\frac{g}{2}\omega_{\mu a}[\tau^a, \gamma^\mu]\psi_{\mathrm{DIR}} - \hat{m}\psi_{\mathrm{DIR}} = 0 \end{aligned} \quad (5.24)$$

Here $\epsilon^{\mu\lambda}$ is defined by (5.1), from which we obtain in the lowest order [see (5.2)] and for a sufficiently small Higgs mass (5.1b) [cf. also (5.6)]

$$\partial^\lambda \epsilon_{\mu\lambda} = 0 \Rightarrow \partial^\lambda \epsilon_{\lambda\mu} = 2\partial^\lambda \epsilon_{[\lambda\mu]} \quad (5.25)$$

Here equation (5.24) takes the final form

$$i\gamma^\mu \partial_\mu \psi_{\mathrm{DIR}} + i[\varepsilon^\lambda{}_\mu \partial_\lambda + (\partial^\lambda \varepsilon_{[\lambda\mu]})]\gamma^\mu \psi_{\mathrm{DIR}} - \frac{g}{2}\omega_{\mu a}\{\tau^a, \gamma^\mu\}\psi_{\mathrm{DIR}} - \hat{m}\psi_{\mathrm{DIR}} = 0 \quad (5.26)$$

where the second term describes the gravitational interaction [in the classical macroscopic limit, cf. (5.8), or already because of (5.6b) the term $\partial^\lambda \epsilon_{[\lambda\mu]}$ vanishes]; the third term represents the interaction with the very massive [cf. (4.7)] gauge bosons.

In its nonrelativistic limit equation (5.26) goes over into the Schrödinger equation with usual Newtonian gravitational potential. Considering the gravitational field as a classical one, equation (5.26) takes the simple form

$$i\gamma^\mu \mathscr{D}_\mu \psi_{\text{DIR}} - \hat{m}\psi_{\text{DIR}} = 0, \qquad \mathscr{D}_\mu = \partial_\mu + \epsilon^\lambda{}_\mu \partial_\lambda \tag{5.27}$$

under neglect of the gauge-boson interaction. Iteration of (5.27), elimination of all spin influences, and linearization in $\epsilon^\lambda{}_\mu$ give with the use of (5.25)[11]

$$(\partial^\mu \partial_\mu + 2\epsilon^{(\nu\mu)} \partial_\nu \partial_\mu)\psi_{\text{DIR}} + \frac{\hat{m}^2 c^2}{\hbar^2} \psi_{\text{DIR}} = 0 \tag{5.28}$$

With the ansatz $(\hat{m}^2 \to m^2)$

$$\psi_{\text{DIR}} = e^{-i(mc^2/\hbar)t} \varphi(x^\nu) \tag{5.29}$$

we obtain from (5.28) under the neglect of all terms up to the order of $c^{-1}(\epsilon^{(\nu\mu)} \sim c^{-2})$ the Schrödinger equation:

$$\frac{\hbar^2}{2m} \Delta\varphi + mc^2 \epsilon^{(00)} \varphi = \frac{\hbar}{i} \partial_t \varphi \tag{5.30}$$

with $\epsilon^{(00)} = -\Phi/c^2$ according to (5.12a), i.e., the usual Schrödinger equation with classical gravitational potential Φ.

We have shown this explicitly, because this quantum mechanical equation has been tested experimentally until now only for the gravitational interaction by the neutron-interference experiment of Collela *et al.* (1975). It may be of interest, however, that the Schrödinger equation (5.30) does not guarantee that atomic clocks and lengths measure the effective non-Euclidian metric; for this the influence of the gravitational field on the electric Coulomb potential between electron and nucleus of the atom is necessary (see, e.g., Papapetrou, 1956), which is not yet included in our theory.

Finally, for the inhomogeneous Yang–Mills equation we obtain from (3.12) and (3.12a) with the use of (4.2), (4.4), (4.6), and (4.8a)

$$\begin{aligned} &\partial_\nu F^{\nu\mu}{}_a + g\epsilon_a{}^{bc} F^{\nu\mu}{}_b \omega_{\nu c} + M^2_{ab} \omega^{\mu b} + 2\epsilon_{(\rho\nu)} M^2_{ab}{}^{\rho\nu} \omega^{\mu b} \\ &\quad = 4\pi \left\{ \frac{g}{2} \bar{\psi}_{\text{DIR}} \{\gamma^\lambda, \tau_a\} \psi_{\text{DIR}} (\delta^\mu{}_\lambda + \epsilon^\mu{}_\lambda) + ig \frac{v^2}{16} (\partial^\mu \epsilon_{\rho\nu}) \operatorname{tr}([\gamma^\rho, \tau_a]\gamma^\nu) \right\} \end{aligned} \tag{5.31}$$

[11] We use $\hbar$ and c explicitly because of the ordering with respect to c^{-1}.

where we have restricted ourselves also to linearized interaction terms with respect to gravitation. On the left-hand side we recognize the mass term and the interaction of the massive bosons with the gravitational potentials; on the right-hand side we find as sources gravitationally influenced Dirac gauge currents and a current associated with the gravitational field itself (remaining Higgs-field current). Because of this it may be justified to call the gauge-boson interaction as a "strong" but very massive gravitational interaction; its coupling constant g remains, however, undetermined within our present theoretical approach.

6. FINAL REMARKS

In extension of a previous spin-gauge theory of gravity we have shown that Dirac's γ-matrices can be treated as a quantizable Higgs field, in consequence of which Einstein's metrical theory of gravitation follows as the classical macroscopic limit of the Higgs-field interaction after symmetry breaking.

In spite of this success there are several problems for the future. First, not only is the effective space-time geometrical structure a Riemannian one, but also nonmetricity is present, which should be suppressed in the next step since no observational hint for it exists. This may be possible because the Lagrange density (3.2) for the Higgs field is not yet unique but can be supplemented in its kinetic term, e.g., by $\mathrm{tr}[(D^{\alpha}\tilde{\gamma}^{\mu})(D_{\mu}\tilde{\gamma}_{\alpha})]$. In connection with this it may also be attainable to avoid the constraint (5.6), which corresponds to the de Donder condition, and perhaps in this way Einstein's theory can be reached even exactly and not only in its linearized version as presented above.

Furthermore the theory, as it stands, contains only the gravitational interaction between the fermions. But the gravitational interaction with all bosons must be included within a complete and consistent theory of gravitation; otherwise, as remarked above (cf. Papapetrou, 1956), atomic clocks and lengths do not measure the non-Euclidian effective metric. This may require, however, a unification with the other interactions on the microscopic level of unitary phase-gauge transformations within a high-dimensional (e.g., eight-dimensional) spin-isospin space.

In this respect one could have a bold idea: Because in our theory the γ-matrices are treated as a Higgs field, it could be possible to introduce the chiral asymmetry of the fermions with regard to the weak interaction, which is, however, present already in the $SU(5)$ GUT, by a special choice of the ground state of the γ-Higgs field in the course of the spontaneous symmetry breaking at approximately 10^{19} GeV connected with the gravitational interaction.

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